

## On a Remarkable Class of Subvarieties of a Symmetric Variety

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### INTRODUCTION

Let  $G/H$  be a symmetric space for the involution  $\theta: G \rightarrow G$ , where  $G$  is a semisimple algebraic connected group of adjoint type, and let  $H := G^\theta$  be the subgroup of fix-points of  $\theta$ .

We denote by  $T_1 \subset G$  a maximal  $\theta$ -split torus (a maximal anisotropic torus), and by  $\mathfrak{t}_1 := \text{Lie } T_1$ ,  $\mathfrak{g} := \text{Lie } G$  the corresponding Lie algebras. Recall that the involution  $\theta$  induces an automorphism of order 2 on  $\mathfrak{g}$ , still denoted by  $\theta$ , which acts on  $\mathfrak{t}_1$  as minus the identity, i.e.,  $\theta y = -y$  for every  $y$  in  $\mathfrak{t}_1$ .

Let  $T \subset T_1$  be a maximal torus in  $G$  and  $W$  the corresponding Weyl group. It is known that  $T$  is  $\theta$ -stable and one can construct the subquotient  $W_1$ , the restricted Weyl group with respect to  $T_1$  (of Section 1.1).

We denote by  $X$  the  $G$ -equivariant smooth compactification of  $G/H$  described by C. De Concini and C. Procesi [4], and we regard the points of  $X$  as subalgebras of the Lie algebra  $\mathfrak{g}$ .

In  $X \times \mathfrak{g}$  we define the subset

$$Z := \{(I_x, z) \text{ such that } z \in I_x^\perp\}_{x \in X} \subset X \times \mathfrak{g},$$

where  $I_x$  is the subalgebra associated to the point  $x \in X$  and  $I_x^\perp$  is its orthogonal complement in  $\mathfrak{g}$  with respect to the Killing form.

We have the natural projection  $\pi: Z \rightarrow \mathfrak{g}$  defined via  $\pi(I_x, z) = z \in \mathfrak{g}$ .

Let  $y \in \mathfrak{g}$  be a semisimple element conjugate to an element in  $\mathfrak{t}_1$ . In particular we will take  $y \in \mathfrak{t}_1$ . In this paper we study the fiber  $\pi^{-1}(y)$  of the projection  $\pi$  and its orbit structure with respect to the action of the stabilizer  $C_y := \text{Stab}_G y$  of the semisimple element  $y \in \mathfrak{t}_1$ . In fact  $\pi^{-1}(y)$  is identified as the subset of the points  $x \in X$  such that the orthogonal  $I_x^\perp$ , of the corresponding subalgebra  $I_x$ , contains the given  $y$ . Therefore the action of  $G$  on the variety  $X$  induces an action of  $C_y$  on  $\pi^{-1}(y)$ .

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Let  $\bar{T}_1$  be the closure of the orbit under  $T_1$  of the class of 1 in  $G/H$ . This is a compactification of the torus  $T_1/T_1 \cap H$ . Note that  $T_1 \cap H = \{x \in T_1 \text{ such that } x^{-1} = x\}$ , i.e., the group of the elements of order 2 in  $T_1$ .

We have the inclusions

$$\bar{T}_1 \subset \pi^{-1}\{y\} \subset X.$$

In fact let  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$  be the decomposition in eigenspaces of  $\mathfrak{g}$  with respect to the involution  $\theta$ .  $\mathfrak{g}_1$  equals  $\text{Lie } H$  so we will denote it by  $\mathfrak{h}$ . We have  $\mathfrak{t}_1 \subset \mathfrak{g}_{-1} = \mathfrak{h}^\perp$  ( $\mathfrak{h}$  and  $\mathfrak{t}_1$  are orthogonal). It follows that  $\mathfrak{h} \subset \pi^{-1}(y)$  for every  $y \in \mathfrak{t}_1$ . Moreover  $C_y \supset T_1$ ; therefore  $T_1/T_1 \cap H = T_1 \mathfrak{h} \subset \overline{C_y} \mathfrak{h} \subset \pi^{-1}(y)$ . As  $\pi^{-1}(y)$  is closed in  $X$  it follows that  $\bar{T}_1 \subset \pi^{-1}(y)$ .

Let  $\mathcal{O}$  be a  $G$ -orbit of the action on  $X$ ; the intersection  $\pi^{-1}(y) \cap \mathcal{O}$  is  $C_y$ -stable. In this paper we prove the following:

**THEOREM.** (i)  $\pi^{-1}(y) \cap \mathcal{O}$  is a union of a finite number of  $C_y$ -orbits, for every  $y \in \mathfrak{t}_1$ .

(ii) Each  $C_y$ -orbit in  $\pi^{-1}(y)$  intersects  $\bar{T}_1$  in a union of  $T_1$ -orbits which are permuted transitively by the group  $W_{1_y} := \text{Stab}_{W_1} y$ , i.e., the stabilizer of  $y$  in the restricted Weyl group  $W_1$ .

(iii) The  $C_y$ -orbits in  $\pi^{-1}(y) \cap \mathcal{O}$  are closed in  $X$  if and only if  $\mathcal{O}$  is the closed orbit in  $X$ .

(iv)  $\pi^{-1}(y)$  is smooth and connected.

(v) The closures of the  $C_y$ -orbits in  $\pi^{-1}(y)$  are smooth and each one is the transverse intersection of the codimension 1 orbit closures that contain it.

The study of the variety  $\pi^{-1}(y)$  is suggested by the work of De Concini, Goresky, MacPherson, and Procesi "On the Geometry of Complete Quadrics" (preprint, IHES, 1986), where the theorem of this paper is proved in the case of the symmetric space of quadrics.

In any case the previous theorem shows that  $\pi^{-1}(y)$  is a very nice compactification of a symmetric variety of a more general type, i.e.,  $C_y/C_y^\theta = C_y/C_y \cap H$ , but now  $C_y$  is no longer semisimple but only reductive.

## 1. PRELIMINARIES

### 1.1. Root System for a Symmetric Space

Let  $G$  be a connected reductive algebraic group defined over an algebraically closed field  $F$  ( $\text{char } F \neq 2$ ), and let  $\theta \in \text{Aut}(G)$  be an involutorial automorphism of order 2, i.e.,  $\theta^2 = \text{id}$ .

Let  $H := G^\theta := \{g \in G \text{ such that } \theta(g) = g\}$  be the set of the points of the group  $G$  which are fixed under  $\theta$ .  $H$  is a closed reductive subgroup of  $G$  [11]. The space  $G/H$  is then a symmetric space (with respect to the involution  $\theta$ ).

A torus  $A \subset G$  is called  $\theta$ -split (or anisotropic) if  $\theta(a) = a^{-1}$  for every  $a \in A$ . If  $\theta \neq \text{id}$  then non-trivial  $\theta$ -split tori exist (cf. [11]), in particular there are maximal  $\theta$ -split tori in  $G$ .

Let  $T_i \subset G$  be a maximal  $\theta$ -split torus and let  $T \supset T_i$  be a maximal torus in  $G$  containing  $T_i$ . We fix these choices once and for all. Let  $\mathfrak{t}_1, \mathfrak{t}, \mathfrak{g}, \mathfrak{h}$  be respectively the Lie algebras of  $T_i, T, G, H$ .

First note that  $\theta$  induces a linear automorphism of  $\mathfrak{g}$ , still denoted by  $\theta$ , and  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$  is the decomposition of  $\mathfrak{g}$  into eigenspaces;  $\mathfrak{g}_1 = \mathfrak{h}$  and  $\mathfrak{g}_{-1} = \mathfrak{h}^\perp$ , where  $\mathfrak{h}^\perp$  is the orthogonal subspace of  $\mathfrak{h}$  with respect to the Killing form. Moreover  $\mathfrak{t}$  is  $\theta$ -stable.

Recall now that the maximal torus  $T$  determines the roots  $\phi := \phi(T)$  for the group  $G$ ; i.e., under the adjoint action of  $T$  on  $\mathfrak{g}$ ,  $\mathfrak{g}$  itself decomposes into eigenspaces, the decomposition being

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in \phi} \mathfrak{g}_\alpha,$$

and  $\phi \subset \mathfrak{t}^*$ .

We denote by

$$W := W(T) = N_G(T)/Z_G(T)$$

the corresponding Weyl group.

Let  $X^*(T)$  denote the additively written group of rational characters of  $T$  and  $X_*(T)$  the group of rational one-parameter multiplicative subgroup of  $T$ . Then  $X^*(T)$  can be put in duality with  $X_*(T)$  by the integral valued pairing  $\langle \cdot, \cdot \rangle$  defined as

$$x \in X^*(T), \ell \in X_*(T), \langle x, \ell \rangle(t) = x(\ell(t)) = t^{\langle x, \ell \rangle}, \text{ for all } t \in F^*.$$

We assume now  $F = \mathbb{C}$  (the complex numbers).

Note that if  $T$  is a maximal torus in  $G$  and  $\phi(T)$  are the corresponding roots, then  $\phi(T)$  gives a root system in the space  $V = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ , using the integral pairing. Moreover there is a canonical isomorphism

$$\mathfrak{t} \sim X_*(T) \otimes_{\mathbb{Z}} \mathbb{C} = V \otimes_{\mathbb{R}} \mathbb{C},$$

compatible with the identification of  $\phi$  as linear forms. In particular  $\mathfrak{t}$  and  $\mathfrak{t}_1$  have canonical real forms, i.e., the vectors  $y$  such that  $\alpha(y)$  is real for all  $\alpha \in \phi$ . The real part of  $\mathfrak{t}$  has been identified with  $V$ .

Consider now the involution  $\theta$  acting on  $G$ . We have an induced action

on  $X^*(T)$ ; let  $X_0^* = \{x \in X^*(T): \theta(x) = x\}$ , and let  $\phi_0 := \phi_0(\theta) = \phi \cap X_0^*$ . It follows that the roots  $\phi$  decompose into

$$\phi = \phi_0 \cup \phi_1,$$

where  $\phi_0$  is the set of  $\theta$ -fix roots.

We can choose now a basis  $A$  for the root system  $\phi$  in such a way that

- (1)  $A_0 := A \cap \phi_0$  is a base for  $\phi_0$ ;
- (2)  $A$  is naturally decomposed,

$$A = A_0 \cup A_1, \quad A_1 = A - A_0;$$

- (3)  $\phi_1 = \phi_1^+ \cup \phi_1^-$ , where  $\theta$  acts on  $\phi_1$  sending the positive roots  $\phi_1^+$  (with respect to  $A$ ) into the negative roots  $\phi_1^-$ .

Note that the decomposition (2) is related to the Satake diagram for the symmetric space  $G/H$  (cf. [10]). Note also that  $\phi_0 \subset \mathfrak{t}^*$  is the set of roots which vanish when restricted to  $\mathfrak{t}_1$  (or to its real form  $V_1$ ). The restriction of the elements of  $\phi_1$  to  $\mathfrak{t}_1$  gives rise to the restricted root system of the symmetric space  $G/H$ . In particular the restriction to  $\mathfrak{t}_1$  of the elements in  $A_0$  vanishes identically on  $\mathfrak{t}_1$ , and the restriction  $r$  to  $\mathfrak{t}_1$  of the elements of  $A_1$  gives rise to a basis  $\bar{A}_1$  for the restricted root system

$$r: A_1 \longrightarrow \bar{A}_1.$$

Let  $W' \subset W$  be the subgroup of the Weyl group  $W$  which leaves  $\mathfrak{t}_1$  invariant as subspace and let  $K \subset W'$  be the subgroup whose elements act on  $\mathfrak{t}_1$  as the identity. The restricted Weyl group of the symmetric space  $G/H$  is the quotient  $W_1 = W'/K$ .

## 1.2. Decomposition into Weyl Chambers and Faces

Let  $\phi$  be the root system of  $G$  relative to  $T$ , as described in Section 1.1. The real space  $V$  is decomposed into Weyl chambers and faces as follows. The fundamental chamber  $C$  is defined by

$$\begin{aligned} C &= \{P \in V: \alpha_i(P) > 0, \text{ for every } \alpha_i \in A\} \\ &= \{\sum a_i \omega_i, a_i > 0, (\omega_i, \alpha_i) = \delta_{ij}, \text{ for every } \alpha_i \in A\}. \end{aligned}$$

The other Weyl chambers are the transformations of  $C$  via the Weyl group  $W$ .

The closure  $\bar{C}$  of the fundamental chamber  $C$  is a union of faces  $F_J$  which correspond to the subsets  $J \subset A$ ; we have in fact

$$\begin{aligned} F_J &= \{P \in V: \alpha_i(P) = 0 \text{ for every } \alpha_i \in J, \alpha_i(P) > 0 \\ &\quad \text{for every } \alpha_i \in A - J\}. \end{aligned}$$

We have  $F_J \subset \bar{C}$  and  $C = F_\emptyset$  corresponds to the empty set. The faces  $wF_J \subset w\bar{C}$ , for  $w \in W$ , decompose the space  $V$ .

The Weyl chambers are a unique orbit under the Weyl group  $W$ . The faces decompose into  $W$ -orbits, each orbit having a unique representative in the closure of the fundamental chamber  $C$  (cf. [2]).

Consider now the maximal  $\theta$ -split torus  $T_1 \subset T \subset G$  we have chosen once and for all, and the decomposition  $\Lambda = \Lambda_0 \cup \Lambda_1$ . Let  $V_1 = V \cap \mathfrak{t}_1$ ;  $\mathfrak{t}_1$  is the complexification of  $V_1$  and  $\bar{\Lambda}_1$  is a basis of the dual of  $V_1$ . Using the restricted root system and the basis  $\bar{\Lambda}_1$  we have a decomposition of  $V_1$  into Weyl chambers and faces which are obtained via the action of the restricted Weyl group  $W_1$  on the faces of the closure of the fundamental chamber  $C_1$ ,

$$C_1 = \{P \in V_1 : \alpha(P) > 0 \text{ for every } \alpha \in \bar{\Lambda}_1\}.$$

The faces in the closure  $\bar{C}_1$  correspond to the subsets  $\bar{J} \subset \bar{\Lambda}_1$  in the usual way. Note also that  $C_1 \subset \bar{C}$  but if  $\Lambda_0 \neq \emptyset$ ,  $C_1 \neq C$ .

We now present some lemmas which will be useful in the following sections.

**LEMMA 1.** *Let  $J \subset \Lambda$  and  $F_J$  be the corresponding face. Then  $F_J \cap V_1 \neq \emptyset$  if and only if  $J = \Lambda_0 \cup r^{-1}(\bar{J})$ , where  $\bar{J} \subset \bar{\Lambda}_1$ . Moreover  $F_J \cap V_1 = F_{\bar{J}}$ , where  $F_{\bar{J}}$  is a face of the fundamental chamber  $C_1$  of  $V_1$  and conversely.*

*Proof.* Let  $P \in F_J \cap V_1$ ; it follows that  $\alpha(P) = 0$  for every  $\alpha \in J$  and  $\alpha(P) > 0$  for every  $\alpha \in \Lambda - J$ . Moreover  $P \in V_1$  and therefore  $\alpha(P) = 0$  for every  $\alpha \in \Lambda_0$ ; it follows that  $\Lambda_0 \subset J$ .

If  $\alpha_1, \alpha_2 \in \Lambda_1$  and  $r(\alpha_1) = r(\alpha_2)$  then  $\alpha_1(P) = \alpha_2(P)$  and  $\alpha_1 \in J$  if and only if  $\alpha_2 \in J$ . It follows that  $J = \Lambda_0 \cup r^{-1}(\bar{J})$ , where  $\bar{J} \subset \bar{\Lambda}_1$  is given by the restricted simple roots which vanish on  $P \in F_J \cap V_1$ . It follows that  $P \in F_{\bar{J}}$ . ■

**LEMMA 2.** *Let  $F_1 \cap V_1 = F'_1$  and  $F_2 \cap V_1 = F'_2$  be faces of the decomposition of  $V_1$ . They are equivalent under the action of  $W_1$  if and only if  $F_1$  and  $F_2$  are equivalent in  $V$  under the action of  $W$ .*

*Proof.* We reduce to the case where  $F_1 \cap V_1$  and  $F_2 \cap V_1$  are faces in the closure of the fundamental chamber  $C_1 \subset V_1$ . In particular  $F_1$  and  $F_2$  are faces of the closure of the fundamental chamber  $C \subset V$ . If  $F_1$  and  $F_2$  are equivalent with respect to  $W$  then  $F_1 = F_2$ , which implies  $F_1 \cap V_1 = F_2 \cap V_1$ . The converse is also clearly true. ■

**LEMMA 3** (cf. [2]). *If  $F_1, F_2$  are two faces for the root system  $\phi$  and  $w, w' \equiv W$  are such that  $wF_1 = w'F_1 = F_2$  then the maps from  $F_1$  to  $F_2$  induced by the two elements  $w$  and  $w'$  coincide.*

LEMMA 4. *For every  $y \in t_1$  the set  $Wy \cap t_1$  is a unique orbit under the action of the restricted Weyl group  $W_1$ .*

*Proof.* Identify  $t_1$  to  $V_1 \otimes_{\mathbb{R}} \mathbb{C}$ . We first assume  $P, Q \in V_1$  and such that  $P = wQ$  for  $w \in W$ .  $P \in F'_1 = F_1 \cap V_1$ ,  $Q \in F'_2 = F_2 \cap V_1$ , where  $F'_1$  and  $F'_2$  are faces of the decomposition of  $V_1$ . It follows that  $F_1$  and  $F_2$  are  $W$ -equivalent and by Lemma 2 they are also  $W'$ -equivalent; therefore there exists an isometry  $w' \in W'$  such that  $P = w'Q$  (cf. Lemma 3) and  $P$  and  $Q$  are equivalent under the restriction of  $w'$  to  $V_1$ .

For the general case we recall that an element  $y \in t_1$  is written in a unique way as  $y = Q_1 + iQ_2$  and the Weyl group acts separately on the real and on the imaginary part. Assume  $(P_1, P_2) = w(Q_1, Q_2)$  for  $w \in W$ . Then  $w(Q_1 + \ell Q_2) = P_1 + \ell P_2$  for every  $\ell \in \mathbb{R}$ . From the previous argument there exists an element  $w'_\ell \in W'$  such that  $w'_\ell(Q_1 + \ell Q_2) = P_1 + \ell P_2$ . As  $W'$  is finite there are two distinct values  $\ell_1$  and  $\ell_2$  such that  $w'_{\ell_1} = w'_{\ell_2} = w'$  and

$$w'(Q_1 + \ell_1 Q_2) = P_1 + \ell_1 P_2, \quad w'(Q_1 + \ell_2 Q_2) = P_1 + \ell_2 P_2.$$

It follows that  $(\ell_1 - \ell_2)(w'Q_2 - P_2) = 0$ , from which we deduce  $w'Q_2 = P_2$  and  $w'Q_1 = P_1$ . ■

LEMMA 5. *Let  $F'_1 = F_1 \cap V_1$ ,  $F'_2 = F_2 \cap V_1$  be faces in  $V_1$  and  $y \in t_1$ ; let  $W_y := \text{Stab}_W y$ ,  $W_{1,y} := \text{Stab}_{W_1} y$ . If  $F_1$  and  $F_2$  are equivalent under  $W_y$  then  $F'_1$  and  $F'_2$  are equivalent under  $W_{1,y}$  and conversely.*

*Proof.* Let  $y = P + iQ$ ,  $P, Q \in V_1$ , and let  $A \in F_1$  and  $B \in F_2$  such that  $wA = B$ ,  $w \in W_y$ . Hence  $w$  sends the triple  $P, Q, A$  into  $P, Q, B$ . An argument similar to the one of Lemma 4 proves that there exists an element  $w' \in W'$  sending  $P, Q, A$  to  $P, Q, B$ . ■

### 1.3. $G$ -Orbits of the Compactification $X$ of $G/H$ and Parabolic subalgebras

We recall briefly some of the results of [4], where the minimal compactification of the symmetric space  $G/H$  is described. It is shown in this paper that  $X$  can be viewed also in the sense of Demazure [6]. Let  $\mathfrak{g}$  and  $\mathfrak{h}$  be the Lie algebras respectively of  $G$  and  $H$  and set  $\dim \mathfrak{g} = n$ ,  $\dim \mathfrak{h} = m$ . For every  $g \in G$  we consider the subgroup  $gHg^{-1}$  and its Lie algebra  $\text{ad}(g)\mathfrak{h} \in G_{m,n}$ , where  $G_{m,n}$  is the Grassmann variety of  $m$ -dimensional subspaces in the  $n$ -dimensional space  $\mathfrak{g}$ . The compactification  $X$  coincides with the closure  $\overline{G\mathfrak{h}} \subset G_{m,n}$  of the orbit  $G\mathfrak{h}$  in  $G_{m,n}$ . The boundary points in  $X$  are Lie subalgebras of  $\mathfrak{g}$  which can be explicitly described.

Moreover for each  $G$ -orbit  $\mathcal{O} \subset X$  we have a projection  $\rho: \mathcal{O} \rightarrow G/P$  from the orbit  $\mathcal{O}$  to a corresponding variety of parabolic subalgebras of  $\mathfrak{g}$ , all of the same type, the correspondence being the one that associates to each Lie subalgebra  $\mathfrak{l} \in \mathcal{O}$  the normalizer of its unipotent radical.

Recall now that, up to conjugacy, the parabolic subalgebras in  $\mathfrak{g}$  correspond to subsets of the simple roots  $\Lambda$  of the root system  $\phi$  (cf. [3]). In particular the  $G$ -orbit  $\mathcal{O} \subset X$  projects, via  $\rho$ , to parabolic subalgebras which, up to conjugacy, correspond to a subset  $J \subset \Lambda$  of the form

$$J = \Lambda_0 \cup r^{-1}(\bar{J}),$$

where  $\bar{J} \subset \bar{\Lambda}_1$ . The parabolic is then explicitly given by

$$\mathfrak{p} := \mathfrak{p}_J = \mathfrak{t} \oplus \sum_{\alpha \in \text{span } J} \mathfrak{g}_\alpha \oplus \sum_{\substack{\alpha \in \phi^+ \\ \alpha \notin \text{span } J}} \mathfrak{g}_\alpha.$$

Note that in the orbit  $\mathcal{O}$  there is a canonical Lie subalgebra  $\mathfrak{l} \in \rho^{-1}(\mathfrak{p})$  which is explicitly given by

$$\mathfrak{l} = \mathfrak{t} \oplus \left( \sum_{\alpha \in \text{span } J} \mathfrak{g}_\alpha \right)^\theta \oplus \sum_{\substack{\alpha \in \phi^+ \\ \alpha \notin \text{span } J}} \mathfrak{g}_\alpha.$$

In particular for the compactification  $X$  there is a unique closed orbit  $\mathcal{O}_0$  which corresponds to the subset  $J = \Lambda_0$ .

On the other hand, as we have seen in Section 1.2, the subsets  $J \subset \Lambda$  of the form  $J = \Lambda_0 \cup r^{-1}(\bar{J})$ ,  $\bar{J} \subset \bar{\Lambda}_1$ , correspond to faces of the decomposition of  $V$  which intersect  $V_1$  in faces.

In  $G/P$  all the parabolic subalgebras are conjugate under  $G$ ; those which contain a given maximal Cartan subalgebra  $\mathfrak{t}$  are conjugate under the Weyl group  $W$ , and they correspond to a  $W$ -orbit of faces in  $V$ . In particular the parabolics corresponding to orbits  $\mathcal{O}$  correspond also to faces intersecting  $V_1$ .

Summarizing, we have the following six sets in canonical bijective correspondence:

- (I)  $G$ -orbits in  $X$ ,
- (II) parabolic subgroups corresponding to subsets  $J = \Lambda_0 \cup r^{-1}(\bar{J})$ ,
- (III) faces of the fundamental chamber  $C$  meeting  $V_1$ ,
- (IV) faces of the fundamental chamber  $C_1$  of  $V_1$ ,
- (V)  $W$ -orbits of faces of  $V$  meeting  $V_1$ ,
- (VI)  $W_1$ -orbits of faces in  $V_1$ .

Each one of these may be identified by the set  $J$  and we will say that it is of parabolic type  $J$  or of parabolic type  $P$  if  $P = P_J$ .

#### 1.4. The Action of the Restricted Weyl Group $W_1$ on the Torus Embedding $T_1$

Recall that  $T_1 \subset G$ ,  $T_1/T_1 \cap H \subset G/H \subset X$  and  $\bar{T}_1$  is the closure of  $T_1/T_1 \cap H$  in  $X$ . Since  $T_1 \cap H$  is isomorphic to  $\mathbb{Z}/(2)^r$ ,  $r = \text{rk } T_1$ , we identify  $X_*(T_1)$  with  $2X_*(T_1/T_1 \cap H)$  and so  $V_1 = X_*(T_1) \otimes_{\mathbb{Z}} \mathbb{R}$  with  $X_*(T_1/T_1 \cap H) \otimes_{\mathbb{Z}} \mathbb{R}$ . From the description of the restricted Weyl group  $W_1$  given by Richardson (cf. [9]) we have

$$W_1 = N_H(T_1)/Z_H(T_1),$$

where

$$N_H(T_1) = \{g \in H \text{ such that } gtg^{-1} \in T_1 \text{ for every } t \in T_1\},$$

$$Z_H(T_1) = \{g \in H \text{ such that } gtg^{-1} = t \text{ for every } t \in T_1\}.$$

Let  $x \in T_1/T_1 \cap H \subset G/H$ , then  $x = t_1 H$ , where  $t_1 \in T_1$ ; consider  $g \in N_H(T_1)$ , and we have

$$gx = gt_1 H = gt_1 g^{-1} g H = gt_1 g^{-1} H \in T_1/T_1 \cap H.$$

Moreover if  $g' \in Z_H(T_1)$ ,  $g'$  acts trivially on  $x \in T_1/T_1 \cap H$ . It follows that  $W_1$  acts on  $T_1/T_1 \cap H$  by conjugation. As the subgroup  $N_H(T_1) \subset G$  acts on  $X$  and stabilizes  $T_1/T_1 \cap H$  it follows that it acts on the closure  $\bar{T}_1$ ; similarly the action of  $Z_H(T_1)$  is trivial. It follows that  $W_1$  acts on the compactification  $\bar{T}_1$ .

In this case we have a finite group  $W_1$  which acts on the torus  $T_1/T_1 \cap H$  as a group of automorphisms and the action extends to its closure which is a torus embedding. In [4] it is proved that  $\bar{T}_1$  is smooth. Using the theory of torus embeddings (cf. [8]), the canonically associated decomposition in simplicial cones of  $V_1 = X_*(T_1/T_1 \cap H) \otimes_{\mathbb{Z}} \mathbb{R}$  is exactly the decomposition into Weyl chambers and faces of the restricted root system. The  $T_1$ -orbits of  $\bar{T}_1$  correspond thus to the faces of the decomposition; such orbits are permuted by  $W_1$  in the same way as faces are.

Each  $G$ -orbit  $\mathcal{O} \subset X$  intersects  $\bar{T}_1$  exactly in a  $W_1$ -orbit of  $T_1$ -orbits. By the correspondence with faces such a  $W_1$ -orbit corresponds canonically to a face in the fundamental chamber  $C_1$ , hence to a subset  $\bar{J} \subset \bar{A}_1$ . The parabolic type corresponding to  $\mathcal{O}$  is the one associated to the subset  $J = A_0 \cup r^{-1}(\bar{J})$ .

## 2

Let  $\mathcal{O}$  be a  $G$ -orbit of the action on  $X$  and  $\rho: \mathcal{O} \rightarrow G/P$  the corresponding fibration to parabolic subalgebras.



Let  $(G/P)^y$  be the set of parabolic subalgebras  $\mathfrak{p}$  of  $\mathfrak{g}$  such that  $y \in \mathfrak{p}$ ; let  $(G/B)^y$  be the variety of Borel subalgebras  $\mathfrak{b}$  of  $\mathfrak{g}$  such that  $y \in \mathfrak{b}$ .

We have a natural map  $\xi: (G/B)^y \rightarrow (G/P)^y$  (induced from the projection  $G/B \rightarrow G/P$ ), which is  $C_y$ -equivariant and surjective. In fact, if  $\mathfrak{p} \in (G/P)^y$ , i.e.,  $\mathfrak{p}$  is a parabolic subalgebra of  $\mathfrak{g}$  of the given type and  $y \in \mathfrak{p}$ , it follows that  $\mathfrak{p}$  contains a Borel subalgebra  $\mathfrak{b} \subset \mathfrak{p}$ , and  $y \in \mathfrak{b}$ , but  $\mathfrak{b}$  is also a Borel subalgebra in  $\mathfrak{g}$ .

Note that since  $y$  is semisimple the stabilizer  $C_y$  of  $y$  in  $G$  is a reductive group and  $C_y \supset T$ . We denote by  $W_y$  the corresponding Weyl group

$$W_y = N_{C_y}(T)/T \subset N_G(T)/T = W.$$

The following is known:

**PROPOSITION 1.**  $(G/B)^y$  is a union of  $|W|/|W_y|$   $C_y$ -orbits, each one isomorphic to the variety of the Borel subalgebras of  $\mathfrak{c}_y := \text{Lie } C_y$ .

*Proof.* We consider the  $C_y$ -equivariant map  $\phi: (G/B)^y \rightarrow C_y/B \cap C_y$  which associates to each Borel subalgebra  $\mathfrak{b}$  of  $\mathfrak{g}$  (with  $y \in \mathfrak{b}$ ) the Borel subalgebra of  $\mathfrak{c}_y$  given by  $\mathfrak{b} \cap \mathfrak{c}_y$ . If  $\mathfrak{b} \in (G/B)^y$  clearly  $\mathfrak{b} \cap \mathfrak{c}_y$  is the Lie algebra of the stabilizer of  $\mathfrak{b}$  in  $C_y$ , hence the orbit of  $\mathfrak{b}$  under  $C_y$  is isomorphic to  $C_y/B \cap C_y$  under the map  $\phi$ . Of course, since  $C_y/B \cap C_y$  is complete, the  $C_y$ -orbits in  $(G/B)^y$  are closed. Moreover the fiber of  $\phi$  is finite; in fact if  $\mathfrak{t}$  is a Cartan subalgebra of  $\mathfrak{g}$  containing  $y \in \mathfrak{t}$ , then  $\mathfrak{t}$  is also a Cartan subalgebra in  $\mathfrak{c}_y$ . The Borel subalgebras in  $\mathfrak{c}_y$  containing  $\mathfrak{t}$  are  $|W_y|$  from the corresponding theory of the Weyl group and correspond to the  $|W_y|$   $T$ -fix points in  $C_y/B \cap C_y$ . On top of them in  $(G/B)^y$  there are exactly  $|W|$   $T$ -fix points (i.e., the Borel subalgebras in  $\mathfrak{g}$  containing  $\mathfrak{t}$ ). It follows that each fiber contains  $|W|/|W_y|$  elements and  $(G/B)^y$  is the union of  $|W|/|W_y|$  copies of  $C_y/B \cap C_y$ . ■

From the surjective  $C_y$ -equivariant map  $\xi: (G/B)^y \rightarrow (G/P)^y$  we deduce the following:

**COROLLARY 2.**  $(G/P)^y$  is a union of finite number of closed  $C_y$ -orbits.

Let us consider now the map

$$\rho': \pi^{-1}(y) \cap \mathcal{O} \longrightarrow (G/P)^y$$

induced from the projection  $\rho$  of the orbit  $\mathcal{O}$  onto  $G/P$ . For each subalgebra  $\mathfrak{l} \in \pi^{-1}(y) \cap \mathcal{O}$  and  $y \in \mathfrak{l}^\perp$  let  $\mathfrak{n} \subset \mathfrak{l}$  be the nilradical of  $\mathfrak{l}$ . It follows that  $\mathfrak{n}^\perp \supset \mathfrak{l}^\perp \ni y$ .  $\mathfrak{n}$  is also the nilradical of the parabolic algebra  $\rho(\mathfrak{l})$  which coincides with  $\mathfrak{n}^\perp$ . It follows that the restriction  $\rho'$  of the projection  $\rho$  takes values in  $(G/P)^y$ .

**PROPOSITION 3.** *The preimage, via  $\rho'$ , of a  $C_y$ -orbit in  $(G/P)^y$  is either empty or a unique  $C_y$ -orbit of  $\pi^{-1}(y) \cap \mathcal{O}$ .*

In order to prove this proposition we need some lemmas.

Let  $p_0$  be a given element in  $(G/P)^y$  and consider the subgroup of  $C_y$  which stabilizes  $p_0$ :  $\text{Stab}_{C_y} p_0$ . Let  $P_0 \subset G$  be the parabolic subgroup such that  $p_0 = \text{Lie } P_0$ .  $g$  acts on the parabolic subalgebras by conjugation and  $P_0$  is the subgroup which stabilizes  $p_0$ . Therefore we have the following:

**LEMMA 4.**  $C_y^0 := \text{Stab}_{P_0} y = C_y \cap P_0 = \text{stab}_{C_y} p_0$ .

Let  $R$  be the solvable radical of  $P_0$  and  $\mathfrak{r} := \text{Lie } R$  the corresponding Lie algebra.  $p_0/\mathfrak{r}$  is semisimple. As  $y \in p_0$  let us denote by  $y'$  the corresponding element in the projection  $p_0 \rightarrow p_0/\mathfrak{r}$ .

**LEMMA 5.** *Let  $C_{y'} := \text{Stab}_{P_0/R} y'$ , and we have  $C_{y'} = C_y^0/C_y^0 \cap R$ .*

*Proof.* We show that, via the map  $P_0 \rightarrow P_0/R$ , the subgroup  $C_y^0$  maps to  $C_{y'}$ . In fact, both groups are connected (cf. [1]) and the statement is true at the level of the corresponding Lie algebras.

We now reduce the study of the map

$$\rho': \pi^{-1}(y) \cap \mathcal{O} \longrightarrow (G/P)^y$$

to the case where  $\mathcal{O}$  is the open orbit  $G/H$  of  $X$ .

In order to do this, for a given  $p_0 \in (G/P)^y$  let  $\rho^{-1}(p_0)$  be the preimage via  $\rho$ . The following facts are a reformulation of analogous statements in [4].  $\rho^{-1}(p_0)$  is a set of Lie subalgebra  $\mathfrak{l}$  in  $p_0$ ; the mapping  $\mathfrak{l} \rightarrow \mathfrak{l}/\mathfrak{l} \cap \mathfrak{r}$  to the Lie subalgebras of  $p_0/\mathfrak{r}$  establishes a  $P_0$ -equivariant isomorphism between  $\rho^{-1}(p_0)$  and a variety  $\mathcal{L}$  of subalgebras of  $p_0/\mathfrak{r}$ ; moreover the action of  $P_0$  factors through the action of  $P_0/R$ , as  $R$  acts trivially on  $p_0/\mathfrak{r}$ .

Recall now that we can define on  $p_0/\mathfrak{r}$  an automorphism  $\theta'$  of order 2 such that the family  $\mathcal{L}$  of algebras of  $p_0/\mathfrak{r}$  is a unique orbit with respect to  $P_0/R$  of the element  $\mathfrak{l}_0 = \mathfrak{l}/\mathfrak{l} \cap \mathfrak{r} = (p_0/\mathfrak{r})^{\theta'}$ . Therefore if  $y' \in p_0/\mathfrak{r}$  is conjugate to an element  $y''$  such that  $\theta' y'' = -y''$  we have reduced our study to the case of the open orbit for the compactification of a suitable symmetric space. Otherwise  $\rho^{-1}(p_0) \cap \pi^{-1}(y)$  is empty.

Using Lemma 5 and the previous argument, in order to prove Proposition 3 we are reduced to proving the following:

**PROPOSITION 6.** *For every  $y \in \mathfrak{t}_1$ ,  $\pi^{-1}(y) \cap G/H$  is a unique  $C_y$ -orbit.*

We postpone the proof of this proposition to the end of this section.

Note that  $\pi^{-1}(y) \cap G/H = (G/H)^y$ , where  $(G/H)^y$  is the set of all Lie algebras  $\mathfrak{l} \subset \mathfrak{g}$  conjugate to  $\mathfrak{h} = \text{Lie } H$  and such that  $y \in \mathfrak{l}^\perp$ .

Before proceeding we recall the following:

**THEOREM** (Konstant and Rallis [7]). *Each semisimple element of  $\mathfrak{h}^\perp$  is  $H$ -conjugate to an element of  $\mathfrak{t}_1$ . Two elements in  $\mathfrak{t}_1$  are  $H$ -conjugate if and only if they are  $W_1$ -conjugate.*

In our setting  $y \in \mathfrak{t}_1 \subset \mathfrak{t}$  we have:

**PROPOSITION 7.** *The  $C_y$ -orbits in  $(G/H)^y = \pi^{-1}(y) \cap G/H$  are in 1-1 correspondence with the  $W_1$ -orbits of  $Wy \cap \mathfrak{t}_1$ .*

*Proof.* Let  $G^{\{y\}} := \{g \in G \text{ such that } g^{-1}y \in \mathfrak{h}^\perp\}$ . It is a subgroup which is  $H$ -stable under right multiplication and contains  $H$ .  $G^{\{y\}}$  is clearly  $(G/H)^y$ . We consider now the morphism  $\chi$  from  $G^{\{y\}}/H$  to the set of the  $W_1$ -orbits of  $Wy \cap \mathfrak{t}_1$ , defined as follows. Let  $g \in G^{\{y\}}$  be a representative of the coset  $gH \subset G^{\{y\}}$ , i.e.,  $g^{-1}y \in \mathfrak{h}^\perp$ . By the theorem of Konstant and Rallis there exists an element  $a \in H$  such that  $ag^{-1}y \in \mathfrak{t}_1 \subset \mathfrak{t}$ , and it follows that  $y$  and  $(ag^{-1})y$  are  $G$ -conjugate and belong to  $\mathfrak{t}$ , i.e., they are  $W$ -conjugate, therefore  $ag^{-1}y \in Wy \cap \mathfrak{t}_1$ . If we take another representative  $g' \in G^{\{y\}}$  of the same coset,  $g' = gk$  for  $k \in H$ , we have  $g'^{-1}y = k^{-1}g^{-1}y \in \mathfrak{h}^\perp$ ; from the theorem of Konstant and Rallis it follows that there exists an element  $b \in H$  such that  $bk^{-1}g^{-1}y = cg^{-1}y \in \mathfrak{t}_1$ , where  $c \in H$ , but  $cg^{-1}y = (ca^{-1})(ag^{-1}y) \in \mathfrak{t}_1$ . Therefore  $cg^{-1}y$  and  $a^{-1}g^{-1}y$  are two semisimple elements of  $\mathfrak{t}_1$  which are  $H$ -conjugate and from the theorem of Konstant and Rallis they are also  $W_1$ -conjugate. It follows that the map which associates to the element  $g \in gH$  the element  $ag^{-1}y \in Wy \cap \mathfrak{t}_1$  sends the other representatives of the class  $gH$  to elements of the orbit of  $ag^{-1}y$  with respect to the restricted Weyl group  $W_1$ . This completes the definition of the morphism  $\chi: (G/H)^y = G^{\{y\}}/H \rightarrow \{W_1\text{-orbits of } Wy \cap \mathfrak{t}_1\}$ .

We prove that

- (1)  $\chi$  is surjective,
- (2) two elements of  $(G/H)^y$  are sent to the same  $W_1$ -orbit if and only if they belong to the same  $C_y$ -orbit.

*Proof of (1).* Let  $gy \in Wy \cap \mathfrak{t}_1$ , from  $gy \in \mathfrak{t}_1 \subset \mathfrak{h}^\perp$  we deduce that  $g^{-1} \in G^{\{y\}}$  and  $g^{-1}H$  is mapped via  $\chi$  to the  $W_1$ -orbit of  $gy$ . ■

*Remark.* Let  $gH, g \in G^{\{y\}}$ , be a class which projects to a  $W_1$ -orbit  $S$  in  $Wy \cap \mathfrak{t}_1$  and let  $z \in S$ . We have seen that also the element  $ag^{-1}y$ , for a suitable  $a \in H$ , belongs to  $S$ , therefore  $z$  and  $ag^{-1}y$  are  $W_1$ -conjugate, and by the theorem of Konstant and Rallis they are  $H$ -conjugate. It follows that there exists an element  $a' \in H$  such that  $z = a'(ag^{-1}y) = h^{-1}g^{-1}y = (gh)^{-1}y$ , where  $h^{-1} = a'a \in H$ . Note that  $gh$  is a representative of the class  $gH$ . It follows that if  $\chi(gH) = S$  for every  $z \in S$  there exists a representative  $gh \in gH$  such that  $(gh)^{-1}y = z$ .

*Proof of (2).* Let  $gH$  and  $g'H$  be two distinct elements in  $G^{\{y\}}/H$  such that  $\chi(gH) = \chi(g'H) = S$  and let  $z \in S$  be a given point. By the previous remark there exist  $g_1 \in gH$  and  $g_2 \in g'H$  such that  $g_1^{-1}y = z$  and  $g_2^{-1}y = z$ , therefore  $g_2 g_1^{-1}y = y$  and  $c = g_2 g_1^{-1} \in C_y$ , i.e.,  $g_2 = cg_1$  and  $g'H = c(gH)$ .

Conversely if  $gH$  and  $g'H$  are two cosets in  $G^{\{y\}}$  which are  $C_y$ -conjugate it follows that there exists an element  $c \in C_y$  such that  $g' = cg$  and  $g'g^{-1}y = cy = y$ , i.e.,  $g^{-1}y = g'^{-1}y$  and  $gH$  and  $g'H$  are sent by  $\chi$  to the same  $W_1$ -orbit of  $W_y \cap \mathfrak{t}_1$ . This completes the proof of Proposition 7. ■

We are now ready to give the proof of Propositions 6 and 3.

*Proof of Proposition 6.* It follows immediately from Proposition 7 and Lemma 4 of Section 1.2.

*Proof of Proposition 3.* It is a consequence of Proposition 6, as we are reduced to the case of the open orbit. ■

**COROLLARY 8.** *For every  $G$ -orbit  $\mathcal{O}$  in  $X$  and for every  $y \in \mathfrak{t}_1$ ,  $\pi^{-1}(y) \cap \mathcal{O}$  decomposes into a finite number of  $C_y$ -orbits.*

*Proof.* It is a consequence of Proposition 3 and Corollary 2. ■

### 3

We go back now to the study of  $\pi^{-1}(y) \cap \mathcal{O}$ , where  $\mathcal{O}$  is a  $G$ -orbit in  $X$ . We know from Corollary 8 of the previous section that it is a union of a finite number of  $C_y$ -orbits.

Moreover  $\mathcal{O}$  determines a parabolic corresponding to a subset  $\bar{J} \subset \bar{A}_1$  (cf. Section 1.3). The corresponding face  $F_{\bar{J}}$  in the closure of the fundamental chamber  $C_1$  of  $V_1$  belongs to a  $W_1$ -orbit of faces which correspond on the other side to  $T_1$ -orbits in  $\bar{T}_1$ . This is precisely the intersection  $\bar{T}_1 \cap \mathcal{O}$  (cf. [5]).

Next we assume  $y = y_0$  generic in  $\mathfrak{t}_1$ , i.e.,  $y_0$  is  $W_1$ -equivalent to a point in the fundamental chamber  $C_1$ .

*Remark.* In the projection  $\rho: \mathcal{O} \rightarrow G/P$  the  $T_1$ -orbits in  $\bar{T}_1 \cap \mathcal{O}$  project to the distinct  $T$ -fix points in the  $W_1$ -orbit of  $\mathfrak{p}_0$ . (The reason is that each  $T_1$ -orbit is a  $C_{y_0}$  orbit ( $y_0$  generic) so each such orbit projects to a closed orbit on which  $T_1$  must act trivially.)

**PROPOSITION 1.** *For  $y_0$  generic we have*

$$\pi^{-1}(y_0) = \bar{T}_1,$$

*i.e., the fiber of  $\pi$  is the compactification of the torus  $T_1/T_1 \cap H$ .*

*Sketch of the Proof.* In this case  $W_{1_{y_0}} = \text{id}$ , as  $y_0$  is generic, and from general theorems (cf. [9]) we have that

$$C_{y_0} = T_1 \ltimes Z_H(T_1).$$

We have already seen that for every  $y \in \mathfrak{t}_1$ ,  $\pi^{-1}(y) \supset \bar{T}_1$  (cf. the Introduction). In this case we want to show that  $\pi^{-1}(y_0) = \bar{T}_1$ .  $C_{y_0}$  acts on  $\bar{T}_1$  and  $Z_H(T_1)$  acts trivially, hence the  $C_{y_0}$ -orbits in  $\bar{T}_1$  are just the  $T_1$ -orbits. We will show soon that these are all and we will do this by intersecting with the  $G$ -orbits in  $X$ . ■

Consider the projection  $\rho': \pi^{-1}(y_0) \cap \mathcal{O} \rightarrow (G/P)^{y_0}$ . Let  $\mathfrak{p} = \text{Lie } P$ ,  $P \supset T$ ,  $\mathfrak{p}$  corresponding to the subset  $J = A_0 \cup r^{-1}(\bar{J})$ , where  $\bar{J} \subset \bar{A}_1$ , and to the face  $F_J$  of the fundamental chamber  $C$  of  $V$ .  $\mathfrak{p}$  is a  $T$ -fix point in  $(G/P)^{y_0}$ . The other  $T$ -fix points in  $(G/P)^{y_0}$  form a unique  $W$ -orbit and they correspond to the  $W$ -orbit of the face  $F_J$ . Not all these faces intersect  $V_1$ ; the ones which do this correspond to a unique orbit with respect to  $W_1$ .

**PROPOSITION 2.** *Let  $\mathfrak{p}_1 \in (G/P)^{y_0}$  be a  $T$ -fix point, then  $\rho'^{-1}(\mathfrak{p}_1)$  is not empty if and only if the face corresponding to  $\mathfrak{p}_1$  intersects  $V_1$  in a face.*

*Proof.*  $\mathfrak{p}_1$  is of the form  $\mathfrak{p}_1 = w\mathfrak{p}$  for  $w \in W$ , i.e.,  $W = gT$  for  $g \in N_G(T)$ , therefore  $\mathfrak{p}_1 = g\mathfrak{p}$ . Moreover

$$\mathfrak{p} = \mathfrak{t} \oplus \sum_{\alpha \in \text{span } J} \mathfrak{g}_\alpha \oplus \sum_{\substack{\alpha \in \phi^+ \\ \alpha \notin \text{span } J}} \mathfrak{g}_\alpha,$$

Let  $\mathfrak{l}$  be the standard Lie algebra which projects, via  $\rho'$ , on  $\mathfrak{p}$  (cf. Section 1.3),

$$\mathfrak{l} = \mathfrak{t} \oplus \left( \sum_{\alpha \in \text{span } J} \mathfrak{g}_\alpha \right)^\theta \oplus \sum_{\substack{\alpha \in \phi^+ \\ \alpha \notin \text{span } J}} \mathfrak{g}_\alpha.$$

We have  $y_0 \in \mathfrak{t}_1 \subset \mathfrak{l}^\perp$  by assumption. To say that there exists a Lie algebra  $\mathfrak{l}_1$  in the orbit  $\mathcal{O}$  which projects to  $\mathfrak{p}_1 = w\mathfrak{p}$  and such that  $y_0 \in \mathfrak{l}_1^\perp$  is equivalent to saying that  $w^{-1}y_0 \in w^{-1}\mathfrak{l}_1^\perp$ , i.e., that in the fiber over  $\mathfrak{p}$  there is a Lie algebra  $\mathfrak{l}'$  such that  $w^{-1}y_0 \in \mathfrak{l}'^\perp$ .

We pass modulo the solvable radical  $R$ . We have the semisimple algebra  $\mathfrak{p}/\mathfrak{r}$  which contains the image  $\mathfrak{l}/\mathfrak{l} \cap \mathfrak{r}$  of  $\mathfrak{l}$ . On  $\mathfrak{p}/\mathfrak{r}$  we have the action induced by  $\theta$ . Note that  $\theta$  does not act on  $\mathfrak{p}$ , but acts on its Levi factor  $\mathcal{L}$  and  $\mathfrak{p}/\mathfrak{r} = \mathcal{L}/\mathfrak{z}(\mathcal{L})$  ( $\mathfrak{z}(\mathcal{L})$  the center of  $\mathcal{L}$ ). If we try to satisfy the condition  $w^{-1}y_0 \in \mathfrak{l}'^\perp$ , the same thing happens modulo the radical, i.e.,  $w^{-1}y_0$  modulo the radical is conjugate with respect to  $P/R$  to an element of  $\mathfrak{t}_1/\mathfrak{r} \cap \mathfrak{t}_1$  and this implies that there exists an element  $w''$  of the Weyl group  $W_P$  of the

parabolic  $P$  which conjugates  $w^{-1}y_0$  (mod the radical) in  $t_1/r \cap t_1$ . The center  $z(\mathcal{L})$  of the Levi factor is the set where the roots of  $J$  vanish, but  $J \supset A_0$ , therefore  $z(\mathcal{L}) \subset t_1$ . It follows that to say that  $w''w^{-1}y_0 \in t_1/t_1 \cap r$  is equivalent to saying that  $w''w^{-1}y_0 \in t_1$ .

We have therefore  $y_0 \in t_1$  and  $w''w^{-1}y_0 \in t_1$ ; as these two elements are conjugate and belong to  $t_1$  they are also  $W_1$ -conjugate. It follows that there exists an element  $w' \in W'$  such that  $w'w''w^{-1}y_0 = y_0$ , i.e.,  $z = w'w''w^{-1} \in \text{Stab}_W y_0 = K = \ker(W' \rightarrow W_1)$ , since  $y_0$  is generic in  $t_1$ . It follows that  $z^{-1}w'w''w^{-1} = 1$ , i.e.,  $z^{-1}w'w'' = w$ , where  $z^{-1} \in W'$ ,  $w' \in W'$ , and  $w'' \in W_P$ . We deduce that  $p_1 = wp = z^{-1}w'p$  and the corresponding face is in the orbit of the restricted Weyl group. ■

*Proof of Proposition 1.* The  $C_{y_0}$ -orbits in  $\pi^{-1}(y_0) \cap \mathcal{O}$  are exactly the  $T_1$ -orbits of the torus embedding  $\bar{T}_1$  intersected with  $\mathcal{O}$ . In fact the image of a  $C_{y_0}$ -orbit in  $(G/P)^{y_0}$  contains a  $T$ -fix point which, by Proposition 2, is in the orbit  $W_1 p$ . But we have seen in the previous remark that each such point is  $C_{y_0}$ -fix and the unique  $C_{y_0}$ -orbit lying over it is the corresponding  $T_1$ -orbit in  $\bar{T}_1$ . ■

Suppose now  $y \in t_1$  non-generic. Again we study the projection  $\rho': \pi^{-1}(y) \cap \mathcal{O} \rightarrow (G/P)^y$ . We have seen that  $(G/P)^y$  decomposes in a finite number of  $C_y$ -orbits (cf. Corollary 2 of Section 2). Let  $Z$  be such an orbit; the  $T$ -fix points in  $Z$  are an orbit with respect to the Weyl group  $W_y = N_{C_y}(T)/T \subset N_G(T)/T = W$ . The  $T$ -fix points in  $G/P$  are a unique orbit with respect to  $W$ . Consider  $\text{Stab}_W p \subset W$ ; this is the subgroup of  $W$  generated by the reflections relative to the simple roots of  $J = A_0 \cup r^{-1}(\bar{J})$ ,  $\bar{J} \subset \bar{A}_1$ , which define  $p$ . Therefore the  $T$ -fix points in  $G/P$  are identified to  $W/\text{Stab}_W p$  and they decompose into  $W_y$ -orbits.

**PROPOSITION 3.** *Let  $Z$  be a  $C_y$ -orbit of  $(G/P)^y$ .  $Z$  has a non-empty preimage in  $\rho'$  if and only if among the faces corresponding to its  $T$ -fix points there is one which intersects  $V_1$ .*

*Proof.* We can repeat the argument of Proposition 2 up to the point where we have the relation  $w'w''w^{-1}y_0 = y_0$ , which now is  $w'w''w^{-1}y = y$  for  $y$  non-generic and  $w' \in W'$ ,  $w'' \in W_P$ . It follows that  $z = w'w''w^{-1} \in \text{Stab}_W y = W_y$ . We deduce that  $w'w'' = zw$  and this element, applied to the parabolic  $p$ , tells us that  $w'w''p = w'p = zw p$ . We read this equality as the claim; in fact  $wp$  is an element in  $Z$  by assumption,  $zw p$  is an element in its  $W_y$ -orbit, and this one coincides with  $w'p$ , i.e., a parabolic conjugate to  $p$  with respect to the Weyl group  $W'$ , i.e., it determines a face which intersects  $V_1$  in a face. ■

We recall now the complete picture with respect to the torus embedding  $\bar{T}_1$ . Let  $\mathcal{O}$  be a  $G$ -orbit in  $X$ .  $\bar{T}_1$  intersects  $\mathcal{O}$  in a finite number of  $T_1$ -orbits,

each of which projects to  $G/P$  in a unique parabolic subalgebra. All these parabolic are conjugate with respect to the restricted Weyl group  $W_1$ .

**PROPOSITION 4.** *Each  $C_y$ -orbit of  $\pi^{-1}(y) \cap \mathcal{O}$  intersects the torus embedding  $\bar{T}_1$  in those orbits which correspond to faces of  $V_1$  of the corresponding  $W_1$ -orbit.*

*Proof.* From Proposition 3 we deduce that for any  $y \in \mathfrak{t}_1$ , the  $C_y$ -orbits of  $\pi^{-1}(y) \cap \mathcal{O}$  are in 1-1 correspondence with the  $W_y$ -orbits of faces of the type associated to the parabolic  $P$  in the projection  $\rho: \mathcal{O} \rightarrow G/P$  which intersect  $V_1$ . By Lemma 5 of Section 1.2, these correspond also to  $W_1$ -orbits of faces of type  $P$  in  $V_1$ . ■

#### 4. THE PROOF OF THE MAIN THEOREM

We go back to our main theorem stated in the Introduction.

*Proof of (i).* It is an immediate consequence of Proposition 3 of Section 2, as in Corollary 2 of the same section we have proved that  $(G/P)^x$  is a finite union of  $C_y$ -orbits. ■

*Proof of (ii).* We checked it in Proposition 4 of Section 3 by intersecting with every  $G$ -orbit  $\mathcal{O}$  of the compactification  $X$ . ■

*Proof of (iii).* If a  $C_y$ -orbit in  $\pi^{-1}(y) \cap \mathcal{O}$  is closed in  $X$ , also its intersection with  $\bar{T}_1$  must be closed in  $\bar{T}_1$ . By (ii) this would imply that this intersection contains a  $T_1$ -fix point in  $\bar{T}_1$  which is then in a closed orbit of  $X$ . Hence  $\mathcal{O}$  must be the closed orbit. Conversely if  $\mathcal{O} = G/P_{A_0}$  is the closed orbit then the projection to the corresponding variety of parabolics is the identity and the claim follows by Corollary 2 of Section 2. ■

*Proof of (iv).* We first prove that  $\pi^{-1}(y)$  is irreducible. By Proposition 6 of Section 2 we have that  $\pi^{-1}(y) \cap G/H$  is a unique  $C_y$ -orbit containing  $T_1/T_1 \cap H$ , hence  $\overline{\pi^{-1}(y) \cap G/H} \subset \pi^{-1}(y)$  and  $\overline{\pi^{-1}(y) \cap G/H}$  is  $C_y$ -stable and contains  $\bar{T}_1$ . Since each  $C_y$ -orbit of  $\pi^{-1}(y)$  intersects  $\bar{T}_1$  it follows that it intersects  $\overline{\pi^{-1}(y) \cap G/H}$ . But a  $C_y$ -orbit meets a  $C_y$ -stable space if and only if it is contained in it. It follows that  $\pi^{-1}(y) \subset \overline{\pi^{-1}(y) \cap G/H}$ , i.e., it is the closure of a  $C_y$ -orbit; since  $C_y$  is irreducible the claim follows.

Next we prove that  $\pi^{-1}(y)$  is smooth. To do so, we will study it in a neighborhood of a fix point of the torus embedding. Let us consider a  $T_1$ -fix point in  $\pi^{-1}(y)$ , a point corresponding to a chamber  $C_1$  in  $V_1$ . From a theorem (cf. [4]) relative to the local structure around such a point we know that there exists an affine open set  $\mathcal{A}$  isomorphic in a canonical way to the product of an affine space  $A_h$ , which is the affine torus embedding

corresponding to the chamber itself, and the unipotent radical of the parabolic corresponding to the point

$$\mathcal{A} \sim A_h \times U.$$

Let  $U_y = U \cap C_y$ . We have  $A_h \times U_y \subset \pi^{-1}(y)$  as  $\pi^{-1}(y)$  contains  $\bar{T}_1$  and  $\pi^{-1}(y)$  is  $C_y$ -stable. We prove that

$$A_h \times U_y = \pi^{-1}(y) \cap \mathcal{A}. \quad (*)$$

The intersection of a  $G$ -orbit  $\mathcal{O}$  with  $\mathcal{A}$  is an orbit under  $T_1 \times U$  and precisely it is the set of points  $(x_1, \dots, x_h, u)$  in  $A_h \times U$ , where a set of coordinate  $x_i$  is non-zero. Hence we have the same description (with  $U$  replaced by  $U_y$ ) for the intersection of the  $G$ -orbit with  $\pi^{-1}(y)$ , hence (iv) will be proved using  $(*)$  at least locally around the given point.

Both members of the equality  $(*)$  are irreducible subvarieties of  $\mathcal{A}$ . To prove that they coincide it is enough to prove that they have the same dimension. We compute the dimension of  $\pi^{-1}(y) \cap G/H$  which is the open part of  $\pi^{-1}(y)$ ; i.e., we compute the dimension of  $\text{Stab}_{C_y} H$ .

We may assume that  $C_1$  is the fundamental chamber in  $V_1$ . Recall that the root system  $\phi$  is decomposed into  $\phi = \phi_0 \cup \phi_1$ , where  $\phi_0$  are the  $\theta$ -fix roots, and  $\phi_1 = \phi_1^- \cup \phi_1^+$ . The Iwasawa decomposition of the Lie algebra  $\mathfrak{g}$  is

$$\mathfrak{g} = \mathfrak{t}_0 \oplus \mathfrak{t}_1 \oplus \sum_{\alpha \in \phi_0} \mathfrak{g}_\alpha \oplus \sum_{\alpha \in \phi_1^+} \mathfrak{g}_\alpha \oplus \sum_{\alpha \in \phi_1^-} \mathbb{C}(x_\alpha + \theta(x_\alpha)).$$

It follows that

$$\mathfrak{h} = \mathfrak{t}_0 \oplus \sum_{\alpha \in \phi_0} \mathfrak{g}_\alpha \oplus \sum_{\alpha \in \phi_1^+} \mathbb{C}(x_\alpha + \theta(x_\alpha))$$

$$\mathfrak{u} := \text{Lie } U = \sum_{\alpha \in \phi_1^+} \mathfrak{g}_\alpha.$$

On the other hand,  $\mathfrak{c}_y = \text{Lie } C_y$  is the subalgebra of  $\mathfrak{g}$  which is annihilated by  $\text{ad}(y)$ . As  $\mathfrak{t}_0 \oplus \mathfrak{t}_1 \oplus \sum_{\alpha \in \phi_0} \mathfrak{g}_\alpha \subset \mathfrak{c}_y$ , we only have to impose

$$\left[ y, \sum_{\alpha \in \phi_1^+} \lambda_\alpha x_\alpha + \sum_{\alpha \in \phi_1^+} \gamma_\alpha (x_\alpha + \theta(x_\alpha)) \right] = 0.$$

It follows that

$$\sum_{\alpha \in \phi_1^+} \lambda_\alpha \alpha(y) x_\alpha + \sum_{\alpha \in \phi_1^+} \gamma_\alpha \alpha(y) (x_\alpha + \theta(x_\alpha)) = 0$$



and by linear independence we get  $\lambda_\alpha = 0$  and  $\gamma_\alpha = 0$  if  $\alpha(y) = 0$ . It follows that

$$\begin{aligned} \mathfrak{c}_y &= \mathfrak{t}_0 \oplus \mathfrak{t}_1 \oplus \sum_{\substack{\alpha \in \phi_0 \\ \alpha(y) = 0}} \mathfrak{g}_\alpha \oplus \sum_{\substack{\alpha \in \phi_1^+ \\ \alpha(y) = 0}} \mathfrak{g}_\alpha \oplus \sum_{\substack{\alpha \in \phi_1^+ \\ \alpha(y) = 0}} \mathbb{C}(x_\alpha + \theta(x_\alpha)) \\ \mathfrak{c}_y \cap \mathfrak{h} &= \mathfrak{t}_0 \oplus \sum_{\alpha \in \phi_0} \mathfrak{g}_\alpha \oplus \sum_{\substack{\alpha \in \phi_1^+ \\ \alpha(y) = 0}} \mathbb{C}(x_\alpha + \theta(x_\alpha)) \end{aligned}$$

$$\dim \mathfrak{c}_y / \mathfrak{c}_y \cap \mathfrak{h} = \dim \mathfrak{t}_1 + \# \{ \alpha \in \phi_1^+ : \alpha(y) = 0 \} = \dim A_h + \dim U_y.$$

Since each closed  $C_y$ -orbit in  $\pi^{-1}(y)$  contains one of these  $T_1$ -fix points, it follows that  $\pi^{-1}(y)$  is smooth (cf. also [4]). ■

*Proof of (v).* By the description of the intersection of a  $G$ -orbit with  $\mathcal{A} = A_h \times U$  it follows that a  $G$ -orbit intersected with  $\pi^{-1}(y) \cap \mathcal{A} = A_h \times U_y$  is of the form  $A_h^{(i_1 \cdots i_s)} \times U_y$ , where  $A_h^{(i_1 \cdots i_s)} = \{x_1, \dots, x_h : x_{i_1} = \cdots = x_{i_s} = 0 \text{ and the remaining term } \neq 0\}$ . This is clearly a  $T_1 \times U_y$ -orbit. So these sets are intersections of  $C_y$ -orbits in  $\pi^{-1}(y)$  with  $\mathcal{A}$ . The picture is exactly the same as in [4], the only difference being that not all  $C_y$ -orbits meet  $\mathcal{A}$  since there are several closed  $C_y$ -orbits. The entire picture from the combinatorial point of view is given by (ii). ■

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